Nonequilibrium potential for a reaction-diffusion model: Critical behavior and decay of extended metastable states

D.H. Zanette,* H.S. Wio,* and R. Deza†

Centro Atómico Bariloche, Comisión Nacional de Energía Atómica and Instituto Balseiro, Universidad Nacional de Cuyo, 8400 San Carlos de Bariloche, Rio Negro, Argentina.

(Received 30 May 1995)

We study a piecewise linear version of a bistable reaction-diffusion model of an electrothermal instability: the Ballast model. Our aim is to analyze the change in the relative stability of nonhomogeneous linearly stable states as some parameters—boundary reflectivity and system size—are varied, as well as the critical-like behavior when some of these states coalesce. The analysis is carried out through the use of the nonequilibrium potential or Lyapunov functional for this system, which also allows us to study the decay of the metastable extended states.

PACS number(s): 05.70.Ln, 47.54.+r, 82.40.Ck

I. INTRODUCTION

Pattern formation in nonequilibrium systems has attracted considerable attention during the last decade and has become a very active research field [1]. Due to the extremely rich variety of nonequilibrium systems that arise in physics, chemistry, and biology, there is a large number of possible mathematical descriptions. However, the reaction-diffusion (RD) approach has been shown to be a very fertile source of models for the description of pattern formation phenomena in natural and social sciences, and has become a kind of paradigm for such studies [2].

Related to pattern formation, boundary conditions (BC's) have been recently shown to play a relevant role in the appearance and stability as well as in the propagation of spatial structures, for one- and two-component systems [3–5]. In a recent paper, we were concerned with the role of BC's in pattern selection, and more particularly with the *global stability* of the resulting structures [6]. The analysis was carried out by exploiting the notion of the *nonequilibrium potential* or *Lyapunov functional* of the system. This kind of approach has not been used in the realm of RD systems because it is usually not possible, insofar as some potential conditions are not fulfilled, to obtain a Lyapunov function for a general problem. However, Graham and collaborators [7] —who have been pioneers in introducing those concepts—have investigated the form of such Lyapunov-like functionals in problems related to spatially extended systems far from equilibrium, described by the complex Ginzburg-Landau equations. Their results indicate the possibility of getting information about such functionals as well as about global stability even though the system does not fulfill the above indicated potential conditions. When the Lyapunov functional exists, such an approach offers an alternative way of confronting a problem that has recently attracted considerably attention, both experimentally and theoretically; namely, the relative stability of the different attractors, corresponding to spatially extended states, and the possibility of transitions among them due to the effect of (thermal) fluctuations [8,9].

The specific model we shall focus on, with a known form of the Lyapunov function, corresponds to a simple one-dimensional, one-component model of an electrothermal instability [10,11] or, more generally, to an analog of a broad class of systems called bistable reaction-diffusion models [12]. The particular, nondimensional form that we shall work with is [3,5,6]

$$\partial_t T = \partial^2_{yy} T - T + T_h \theta(T - T_c). \tag{1}$$

In Ref. [6], we have analyzed the global stability of the patterns and the relative change in stability for this model, as some BC's parameter was changed. A similar analysis was also carried out for a particular activator-inhibitor model in Ref. [13]. Here we want to analyze how those results depend on the system size, and discuss the application of some recent studies on the decay of nonhomogeneous states to determine the mean lifetime of these metastable states [14]. We also study the special features that resemble the critical point behavior in equilibrium phase transitions when—as the size or other system parameters are varied—some of the linearly stable structures coalesce.

II. NONEQUILIBRIUM POTENTIAL

For the sake of concreteness, we consider here a class of stationary structures $T(y)$ in the bounded domain $y \in (-y_L, y_L)$ with albedo boundary conditions at both ends,

$$\left.\frac{dT}{dy}\right|_{y=\pm y_L} = \mp k T(\pm y_L), \tag{2}$$

*Also at Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina.

†Permanent address: Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad Nacional de Mar del Plata, Deán Funes 3350, 7600 Mar del Plata, Argentina.

where $k > 0$ is the albedo parameter. These are the spatially symmetric solutions to Eq. (1) already studied in Ref. [3]. Such structures can also be seen as a symmetrization of a set of stationary solutions to the Ballast reaction-diffusion model in the interval $(0, y_L)$ with a Neumann boundary condition at $y = 0$, namely, $dT/dy|_{y=0} = 0$.

We quote here, for reference, the explicit form of the stationary structures:

$$T(y) = T_h \times \begin{cases} \sinh(y_c)\gamma'(k, y_L + y)/\gamma(k, y_L), & -y_L \le y \le -y_c \\ 1 - \cosh(y)\gamma(k, y_L - y_c)/\gamma(k, y_L), & -y_c \le y \le y_c \\ \sinh(y_c)\gamma'(k, y_L - y)/\gamma(k, y_L), & y_c \le y \le y_L, \end{cases} \tag{3}$$

with $\gamma(k, y) = \sinh(y) + k\cosh(y)$, and $\gamma' = \partial_y \gamma$. The double-valued coordinate y_c, at which $T = T_c$, is given by

$$y_c^\pm = \frac{1}{2}y_L - \frac{1}{2}\ln\left[\frac{z\gamma(k, y_L) \pm \sqrt{z^2\gamma(k, y_L)^2 + 1 - k^2}}{1 + k}\right], \tag{4}$$

with $z = 1 - 2T_c/T_h$ $(-1 < z < 1)$.

When $y_c^\pm$ exists and $y_c^\pm < y_L$, the solution (3) represents a structure with a central hot zone $(T > T_c)$ and two lateral cold regions $(T < T_c)$. For each parameter set there are two stationary solutions, given by the two values of y_c. In Ref. [3], it has been shown that the structure with the smallest hot region is unstable, whereas the other one is linearly stable. The trivial homogeneous solution $T = 0$ exists for any parameter set and is always linearly stable. These two linearly stable solutions are the only stable stationary structures under the indicated albedo boundary conditions. Other stationary structures such as periodic solutions are always unstable [3,11]. Therefore, under suitable conditions, we have a bistable situation in which two stable solutions coexist, one of them corresponding to a cold-hot-cold (CHC) structure and the other one to the homogeneous trivial state. The unstable solution is always a CHC structure, with a relatively small hot region.

For given values of z and y_L, the albedo parameter k has to be bigger than unity for both CHC structures to exist. In fact, for $k < 1$ the stable solution is a purely hot structure, with $T(y) > T_c$ for all y. However, for sufficiently high k, $y_c^\pm$ becomes complex and the stationary solutions (3) cease to exist. As for the dependence on the system size, for fixed z and $k > 1$, the two solutions exist for large y_L only. In the following, we concentrate on the region of parameters where the two CHC structures do exist.

For the albedo symmetric solution we are considering here, the Lyapunov functional (LF) reads [6]

$$\mathcal{F}[T] = 2\int_0^{y_L} \left\{-\mathcal{G}[T] + \frac{1}{2}(\partial_y T)^2\right\} dy + kT(y_L)^2, \tag{5}$$

with

$$\mathcal{G}[T] = \int_0^T [-T' + T_h\theta(T' - T_c)]\ dT'. \tag{6}$$

Replacing Eq. (3), we obtain the explicit expression

$$\mathcal{F}^\pm = -T_h^2 y_c^\pm z + T_h^2 \sinh(y_c^\pm)\frac{\gamma(k, y_L - y_c^\pm)}{\gamma(k, y_L)}. \tag{7}$$

For the homogeneous trivial solution $T(y) = 0$, instead, we have $\mathcal{F} = 0$.

In Fig. 1 we have plotted the Lyapunov functional $\mathcal{F}[T]$ as a function of k for a fixed system size, $y_L = 2$, and various values of the ratio T_c/T_h, i.e., for different values of z. The curves correspond to the inhomogeneous structures $\mathcal{F}^\pm$, whereas the horizontal line stands for the LF of the trivial solution. In the bistable zone, $k > 1$, the upper branch of each curve is the LF of the unstable structure, where $\mathcal{F}$ attains a maximum. At the lower branch and for $T = 0$, the LF has a local minimum. For each value of T_c/T_h the curve exists up to a certain critical value k_0, at which both branches collapse. At this point,

$$z^2\gamma(k_0, y_L)^2 + 1 - k_0^2 = 0 \tag{8}$$

[compare with Eq. (4)], and

$$\mathcal{F} \to \mathcal{F}_0 = \frac{T_h^2}{2}\left[\frac{\gamma'(k_0, y_L)}{\gamma(k_0, y_L)} - y_L z + z\ \ln\left(\frac{z\gamma(k_0, y_L)}{1 + k_0}\right)\right]. \tag{9}$$

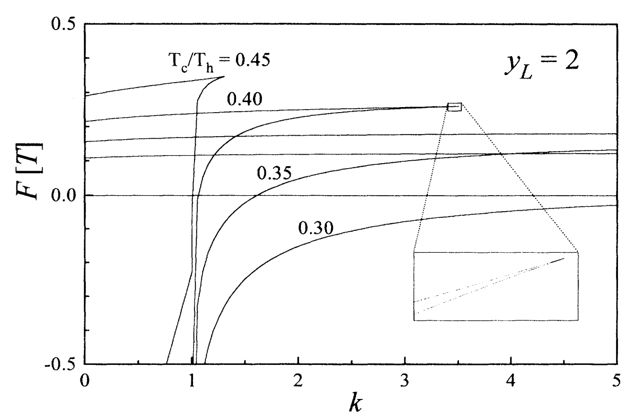

FIG. 1. Values of the nonequilibrium potential $\mathcal{F}$ as a function of the albedo parameter k, for a fixed length ($y_L = 2$) and several values of T_c/T_h. The inset shows the cusplike singularity that occurs when the stable and the unstable CHC structures coalesce and disappear.

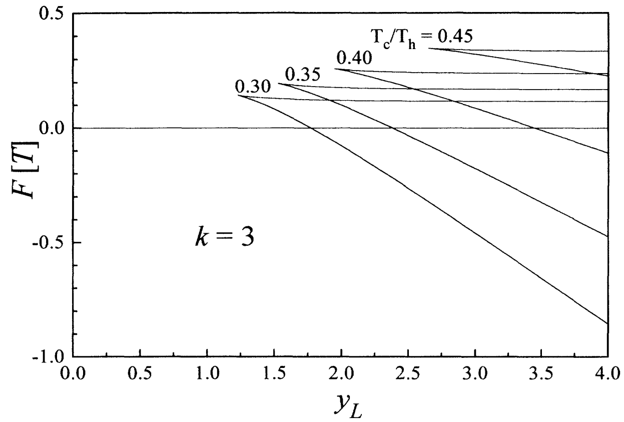

FIG. 2. Values of the nonequilibrium potential $\mathcal{F}$ as a function of the length y_L, for a fixed albedo parameter ($k = 3$) and several values of T_c/T_h.

It is interesting to note that, since the LF for the unstable solution is always positive and, for the stable CHC structure, $\mathcal{F} \to -\infty$ as $k \to 1$, the LF for this structure vanishes for an intermediate value of the albedo parameter. At that point, the stable inhomogeneous structure and the trivial solution interchange their relative stability. In fact, $T(y) = 0$ switches from being a metastable state to be more stable than the inhomogeneous structure.

Figure 2 shows the LF as a function of the system size, for fixed k. Now the inhomogeneous structures exist above a certain value of y_L. For increasing y_L, the trivial solution becomes a metastable state with respect to the stable CHC structure. As $y_L \to \infty$, the LF for this latter solution tends to $-\infty$ as $\mathcal{F} \sim -T_h^2 y_L z/2$. For the unstable solution, instead, the LF approaches a constant, $\mathcal{F} \to T_h^2[z \ln z + (1-z)/\sqrt{z}]/2$.

III. DECAY OF NONHOMOGENEOUS METASTABLE STATES

As indicated in the Introduction, it is of particular interest in RD systems to study the effect of the fluctuations induced by external noise, because they can produce transitions between the different metastable states. The linearly stable states correspond to attractors (minima) of the LF while the unstable ones are saddle points, defining the barrier between attractors [6]. The decay of metastable states has been extensively studied [15]. However, the study of the decay of metastable states in excitable RD and related systems is more recent and scarce [8,9,16].

In order to account for the effect of fluctuations in our model, we need to include in our time-evolution equation (1) a fluctuation term, modeled as an additive noise source [10], yielding a stochastic partial differential equation for the random field $T(y,t)$:

$$\partial_t T(y,t) = \partial_{yy}^2 T - T + T_h \theta(T - T_c) + \xi(y,t). \qquad (10)$$

The simplest assumption about the fluctuation term $\xi(y,t)$ is that it is a Gaussian white noise with zero mean value and a correlation function given by

$$\langle \xi(y,t)\, \xi(y',t') \rangle = 2\,\gamma\, \delta(t-t')\, \delta(y-y'), \qquad (11)$$

where γ denotes the noise strength. It is also possible to take into account noise sources yielding a multiplicative noise term, but we shall not consider this possibility here.

The following scheme has been recently developed in order to describe the decay of extended unstable states [14]. To apply it, we need to assume that the noise strength is weak enough, which assures that the stability of the patterns without noise is qualitatively not altered.

To obtain the transition probability between metastable and stable states, it is necessary to find the conditional probability for the random field $T(y,t)$ to have the value $T_{stable}(y,t)$ at time t, given that at the initial time $t = 0$ the system was in a state $T_{meta}(y,0)$. This probability can be represented by a path integral [17] over those realizations of the random field $\xi(y,t)$ that satisfy the initial and final conditions, that is,

$$P[T_{stable}(y,t)|T(y,0)] \sim \int \mathcal{P}[\xi]\, \delta[T(y,t) - T(y,0)]\, \mathcal{D}\xi(y,t), \qquad (12)$$

where $T(y,0) = T_{meta}(y)$, and the statistical weight $\mathcal{P}[\xi]$ for a Gaussian white noise is of the form

$$\mathcal{P}[\xi] \sim \exp\left[-\frac{1}{2\gamma}\int_0^t dt \int_{-y_L}^{y_L} dy\, \xi^2(y,t)\right]. \qquad (13)$$

In the limit of small noise intensity ($\gamma \to 0$), the main contribution in Eq. (12) is given by the realizations of the random field close to the most probable trajectory [7,14,17]. This fact allows us to estimate the result of Eq. (12) by the steepest-descent method. This procedure, developed in Ref. [14], yields the following result for the mean lifetime (or first-passage time) $\langle\tau\rangle$:

$$\langle \tau \rangle = \tau_0 \exp\left\{\frac{\mathcal{F}[T_{unst}(y)] - \mathcal{F}[T_{meta}(y)]}{\gamma}\right\}. \qquad (14)$$

The factor τ_0 is usually determined by the curvature of $\mathcal{F}[T]$ at its extrema and is typically several orders of magnitude smaller compared to the average time $\langle\tau\rangle$ [15,18].

The behavior of $\langle\tau\rangle$ as a function of the albedo parameter k is shown in Fig. 3, for a system of length $y_L = 2$ and several values of T_c/T_h (corresponding to those depicted in Fig. 1). There is a radical change in the behavior when k overcomes a threshold value, as indicated after Eq. (9), due to the change in the relative stability between the homogeneous and nonhomogeneous states. In Fig. 4 we show the behavior of $\langle\tau\rangle$ as a function of the system length, for a fixed value of the albedo parameter ($k = 3$) and several values of T_c/T_h, this time related to those cases depicted in Fig. 2. Again we witness a change in the behavior due to the change in the relative stability as the system length is varied. The continuous lines in both figures refer to the decay of the metastable state towards the absolutely stable one, while the dotted

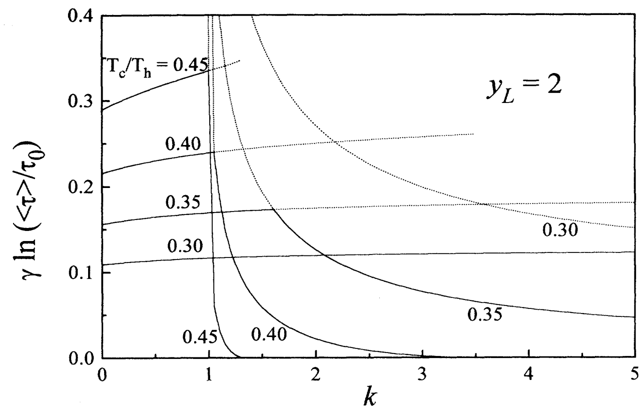

FIG. 3. Values of $\ln(\frac{\langle\tau\rangle}{\tau_0})$ as a function of the albedo parameter k, for a fixed length ($y_L = 2$) and several values of T_c/T_h. The full lines indicate the decay from the unstable to the stable state, while the dotted lines indicate the transition when the relative stability is changed.

line indicates the continuation of the lines depicting the value of $\langle\tau\rangle$ from the original state.

The results just obtained will be valid as long as the barrier height between the metastable and the stable states (given by the value of the LF at the unstable state) is large enough, assuring that the Kramers-like formula Eq. (14) applies. Hence it is relevant to study the system at the points where the indicated barrier can disappear.

IV. CRITICAL-LIKE BEHAVIOR

In this section we study the transition from the bistable to the monostable situation, where the stable and the unstable CHC structures coalesce and disappear. This happens at the point where the LF has the value $\mathcal{F}_0$ indicated in Eq. (9), when the relevant parameters satisfy Eq. (8).

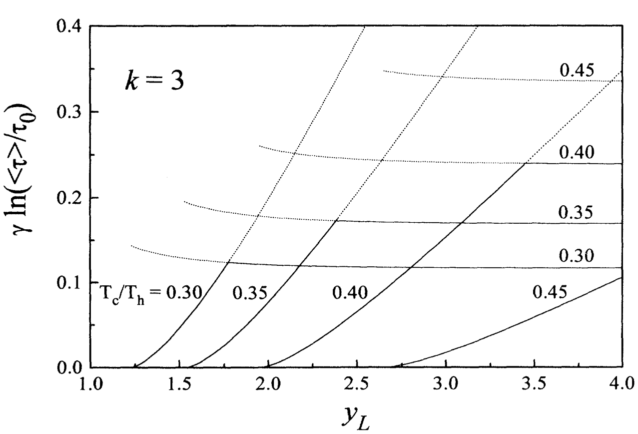

FIG. 4. Values of $\ln(\frac{\langle\tau\rangle}{\tau_0})$ as a function of the length y_L and for a fixed albedo parameter $k = 3$ and several values of T_c/T_h. The full lines indicate the decay from the unstable to the stable state, while the dotted lines indicate the transition when the relative stability is changed.

A detailed analysis of $\mathcal{F}$ near this critical value seems to indicate that a cusp singularity occurs at that point (see inset in Fig. 1). This suggests that near the transition $\mathcal{F}[T]$ admits a phenomenological representation like that of Landau—as a thermodynamical-like potential—by means of a function of the form

$$\mathcal{F}[T] \equiv \mathcal{F}(\eta) = \frac{a}{4}\eta^4 + \frac{b}{3}\eta^3 + \frac{c}{2}\eta^2. \qquad (15)$$

The variable η parametrizes a one-dimensional set in the space of functions $T(y)$ and, in analogy with phase transition problems, plays the role of an *order parameter.* As defined in Eq. (15), $\mathcal{F}$ has a minimum for $\eta = 0$, which we associate with the minimum of the LF at $T(y) = 0$. The other two extrema of $\mathcal{F}(\eta)$ occur at

$$\eta^{\pm} = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}, \qquad (16)$$

where η^- is a maximum—to be associated with the unstable solution—and η^+ is again a minimum, related to the stable CHC structure.

In order to relate the coefficients in Eq. (15) to the original parameters of our problem, we expand $\mathcal{F}$ as given by Eq. (7) around the cusp and identify this critical point with the set of parameters (a, b, c) at which η^+ and η^- merge and disappear. We obtain the equations

$$\frac{p}{6}(1 - 4q)^{3/2} = \frac{[z^2\gamma(k_0, y_L)^2 + 1 - k^2]^{3/2}}{z^2\gamma(k_0, y_L)^3} \qquad (17)$$

and

$$\frac{p}{4}\left(-\frac{1}{6} + q - q^2\right) = \mathcal{F}_0, \qquad (18)$$

for the unknown quantities p and q, which are combinations of the parameters in Eq. (15):

$$p = \frac{b^4}{a^3}, \qquad q = \frac{ac}{b^2}. \qquad (19)$$

Evidently, since in the comparison of Eqs. (7) and (15), the parameters a, b, and c appear through the combinations p and q, only two of them can be obtained independently. This is essentially due to the arbitrariness in the definition of the variable η in the space of functions $T(y)$. In particular, it can be seen that the combinations p and q are invariant under an arbitrary rescaling of η.

The important conclusion to be drawn from this identification of the Lyapunov exponent with a "thermodynamical" function like (15) is that near the critical point at which two of its extrema merge and disappear the problem admits a one-dimensional representation. This feature is in contrast with the infinite-dimensional character of the whole function space, and can be used to strongly simplify the analysis of our system around that critical point. The above indicated mean-field-like approach can be further exploited by means of more elaborate procedures, such as the renormalization group techniques, rendering a more complete description of this critical phenomenon.

V. FINAL REMARKS

Through the knowledge of the nonequilibrium potential or Lyapunov functional, we have studied a piecewise linear reaction-diffusion model representing a bistable system with one-dimensional geometry and partially reflecting boundary conditions.

Such a nonequilibrium potential has allowed us to analyze the global stability of the system and the change in the relative stability between attractors as some parameter (albedo or partial reflectivity at the borders, and/or the system length) is varied. Through this Lyapunov functional, we have also computed the mean lifetime or mean first-passage time for the decay of the metastable stationary state [14]. In this way, we have shown that albedo BC's not only control the relative stability between attractors, but also the response of the system under the effect of fluctuations. We stress that the existence of nontrivial stationary solutions is a consequence of the BC's and the finiteness of the system. On the contrary, for an infinite system, we only have the trivial homogeneous structures as stationary solutions, all nonhomogeneous solutions being unstable.

The kind of calculation of mean first-passage time we have performed here, which is a version for spatially extended systems analogous to the well known Kramers procedure [15], is only valid when such mean first-passage time is much longer than other characteristic times of the system. For this to happen it is necessary that the noise be small enough and the barrier between attractors relatively high. That means we must be far from any critical-like behavior.

We have also shown that for a certain region of parameter values the system behaves in a way resembling an equilibrium phase transition near a critical point. In such a situation, we have obtained a simplified description of the system, that in spite of the infinite-dimensional character of the functional space adopts the form of a kind of one-order-parameter "thermodynamical" potential. The parameters of this "thermodynamical" potential are related to the parameters of the original (infinite-dimensional) nonequilibrium potential. As one might expect, in such a situation the result is analogous to a mean-field description in equilibrium phase transitions. This mean-field description could be improved by more elaborate procedures.

The exploitation of the nonequilibrium potential notion in order to describe the approach to equilibrium within this model when starting from an arbitrary initial condition is under way [19].

ACKNOWLEDGMENTS

The authors thank V. Grunfeld for a critical reading of the manuscript, and R. Graham, E. Tirapegui, and G. Izús for fruitful discussions. Partial support from CONICET, Argentina, through the Grant No. PID 3366/92, and from Fundación Antorchas, Argentina, is also acknowledged.

[1] M.C. Cross and P.C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993); E. Meron, Phys. Rep. **218**, 1 (1992).

[2] G. Nicolis and I. Prigogine, *Self-organization in Nonequilibrium Systems* (Wiley, New York, 1976); P.C. Fife, *Mathematical Aspects of Reacting and Diffusing Systems*, Lecture Notes in Biomathematics Vol. 28, edited by S. Levin (Springer-Verlag, Berlin, 1979); in *Nonequilibrium Cooperative Phenomena in Physics and Related Fields*, edited by M.G. Velarde (Plenum, New York, 1984), p. 371; G. Nicolis, T. Erneux, and M. Herschkowitz-Kaufman, in *Advances in Chemical Physics*, edited by I. Prigogine and S. Rice (Wiley, New York, 1978), Vol. 38; J.D. Murray, *Mathematical Biology* (Springer-Verlag, Berlin, 1985); C. Vidal, and A.V. Pacault, in *Evolution of Order and Chaos in Physics, Chemistry and Biology*, Synergetics Series Vol. 17, edited by H. Haken (Springer-Verlag, Berlin, 1982); J.P. Keener, SIAM J. Appl. Math. **39**, 528 (1980); A.T. Winfree, in *Theoretical Chemistry*, edited by H. Eyring and D. Henderson (Academic Press, New York, 1978).

[3] C.L. Schat and H.S. Wio, Physica A **180**, 295 (1992).

[4] H.S. Wio, G.G. Izús, J.O. Ramírez, R.R. Deza, and C. Borzi, J. Phys. A **26**, 4281 (1993).

[5] S.A. Hassan, M.N. Kuperman, H.S. Wio, and D.H. Zanette, Physica A **206**, 380 (1994); S.A. Hassan, D.H. Zanette, and H.S. Wio, J. Phys. A **27**, 5129 (1994); S.A. Hassan and D.H. Zanette, Physica A **214**, 435 (1995).

[6] G. Izús, R. Deza, O. Ramírez, H. Wio, D. Zanette, and C. Borzi, Phys. Rev. E **52**, 129 (1995).

[7] R. Graham, *Stochastic Processes in Non-equilibrium Systems*, Lecture Notes in Physics Vol. 84 (Springer-Verlag, Berlin, 1978); R. Graham and T. Tel, Phys. Rev. A **42**, 4661 (1990); R. Graham, in *Instabilities and Nonequilibrium Structures*, edited by E. Tirapegui and D. Villaroel (D. Reidel, Dordrecht, 1987).

[8] M. Kerszberg, Phys. Rev. A **28**, 1198 (1983); M. de la Torre and I. Rehberg, *ibid.* **42**, 2096 (1990); **42**, 5998 (1990).

[9] J. Viñals, E. Hernandez-García, M. San Miguel, and R. Toral, Phys. Rev. A **44**, 1123 (1991); E. Hernandez-García, J. Viñals, R. Toral, and M. San Miguel, Phys. Rev. Lett. **70**, 3576 (1993).

[10] H.S. Wio, *An Introduction to Stochastic Processes and Nonequilibrium Statistical Physics* (World Scientific, Singapore, 1994), Chap. 5.

[11] W.J. Skocpol, M.R. Beasley, and M. Tinkham, J. Appl. Phys. **45**, 4054 (1974); B. Ross and J.D. Lister, Phys. Rev. A **15**, 1246 (1977); R. Landauer, *ibid.* **15**, 2117 (1977); D. Bedeaux, P. Mazur, and R.A. Pasmanter, Physica A **86**, 355 (1977); D. Bedeaux and P. Mazur, *ibid.* **105**, 1 (1981).

[12] A.S. Mikhailov, *Foundations of Synergetics I* (Springer-Verlag, Berlin, 1990).

[13] G. Drazer, M.Sc. Thesis, Instituto Balseiro, 1994; G. Drazer and H.S. Wio (unpublished).

[14] A. Foster and A.S. Mikhailov, Phys. Lett. A **126**, 459 (1988); S.P. Fedotov, *ibid.* **176**, 220 (1993).

[15] P. Hängi, P. Talkner, and M. Borkovec, Rev. Mod. Phys. **62**, 251 (1990).

[16] T. Ohta, Prog. Theor. Phys. **99**, 33 (1989); T. Ohta, A. Ito, and A. Tetsuka, Phys. Rev. A **42**, 3225 (1990).

[17] F. Langouche, D. Roekaerts, and E. Tirapegui, *Functional Integration and Semiclassical Expansions* (D. Reidel, Dordrecht, 1982); H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics*, (World Scientific, Singapore, 1990).

[18] J.D. Gunton and M. Droz, *Introduction to the Theory of Metastable and Unstable States*, Lecture Notes in Physics Vol. 183 (Springer-Verlag, Berlin, 1983).

[19] G. Izús, B. von Haeften, R. Deza, H. Wio, and D. Zanette (unpublished).